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Truss Topology Optimization with Simultaneous Analysis and Design

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Strategies for topology optimization of trusses for minimum weight subject to stress and displacement constraints by simultaneous analysis and design (SAND) are considered. The ground structure approach is used. A penalty function formulation of SAND is compared with an augmented Lagrangian formulation. The efficiency of SAND in handling combinations of general constraints is tested. A strategy for obtaining an optimal topology by minimizing the compliance of the truss is compared with a direct weight minimization solution to satisfy stress and displacement constraints. It is shown that for some problems, starting from the ground structure and using SAND is better than starting from a minimum compliance topology design and optimizing only the cross sections for minimum weight under stress and displacement constraints. A member elimination strategy to save CPU time is discussed.

Introduction

THE optimization of the geometry and topology of structural layout has been gaining popularity in recent years, with the development of new methods such as the homogenization method.[1] Traditionally, truss topology optimization problems have been formulated in terms of member forces and by ignoring compatibility requirements to obtain a linear programming (LP) problem in member areas and forces.[2] The solution to this LP problem is then used as a starting point for other formulations.[3] On the other hand, when displacement-based formulations are used, nonzero lower bounds on the cross-sectional areas have been used to guarantee a nonsingular stiffness matrix.[4] A common strategy in truss topology design is the ground structure approach due to Dorn et al.[5] This means that for a given layout of nodal points the optimum topology is obtained as a subset of the initial design which connects each node to every other node. Figure 1 shows an example of a ground structure. Without nonzero minimum area constraints the optimization process will reduce most of the areas to zero, and this means that the stiffness matrix of the truss can become singular and the optimization problem can become nondifferentiable. One way of tackling this problem is to use specialized algorithms for compliance minimization as discussed in Refs. 6 and 7. Another way is to use the simultaneous analysis and design (SAND)[8] approach. SAND treats the displacement as additional design variables and the equations of equilibrium as equality constraints, and it does not require inversion or factorization of the stiffness matrix. Since very efficient algorithms are available for compliance minimization, it is tempting to use them to find the optimal topology and then employ sizing minimization to optimize the cross-sectional areas for the actual design requirements. One objective of the present work is to compare this two-phase strategy with the direct use of the SAND method for problems of minimum weight design subject to stress and displacement constraints. A second objective is to compare a penalty function (PF) formulation of SAND with an augmented Lagrangian (AL) formulation.

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Analysis and Optimization

The SAND approach is a natural way to avoid the nondifferentiability problem associated with a singular stiffness matrix occurring in a truss topology optimization problem due to vanishing member sizes. SAND treats the equations of equilibrium as equality constraints with the nodal displacements used as design variables in addition to the cross-sectional areas of truss members. Since we do not solve the equilibrium equations directly, the global stiffness matrix need not be assembled and factored. Hence, singularities in the stiffness matrix do not pose a problem. The SAND method generally increases the number of design variables substantially, but for truss topology problems this is less of a problem since the ground structure approach leads to a very large number of cross-sectional area design varaibles and comparatively fewer displacement design variables. For example, for the ground structure shown in Fig. 1, the 196 cross-sectional areas are augmented by only 40 displacement variables.

The compliance minimization problem in Ref. 7 is to minimize the compliance $f^T u$ of the truss for a given volume V, where f and u are the force and displacement vectors, respectively. Denoting the elemental volumes as x_i, the problem is formulated as

$$\begin{aligned} &\underset{x,\,u}{\text{minimize}} && f^T u \\ &\text{subject to} && g_j(x, u) = x_j/x_0 \ge 0, \qquad j = 1, \dots, m \\ & && g_{m+1}(x, u) = 1 - \sum_{i=1}^{m} x_i/V \ge 0 \\ & && R = \sum_{i=1}^{m} x_i K_i u - f = 0 \end{aligned} \tag{1}$$

where x_0 is a reference element volume, K_i the stiffness matrix per unit volume of the ith truss element, and R the vector of residuals associated with the equilibrium equations.

The problem solved in the present work is to minimize the volume (hence the weight) of a truss subjected to stress and displacement constraints. The problem is formulated as

$$\begin{aligned} &\text{minimize} && V(x) \\ &\text{subject to} && g_j(x, u) \ge 0, \qquad j = 1, \dots, m \\ & && R = Ku - f = 0 \end{aligned} \tag{2}$$

where K is the stiffness matrix of the structure, and g_j are stress or displacement constraints.

Table 1 Comparison of SAND-based algorithms for aspect ratio 2:1 problem

Problem size	Nondimensional compliance			IBM 3090 CPU time, s		
	Compliance minimization	Weight minimization		Compliance minimization	Weight minimization	
	PF	PF	AL	PF	PF	AL
4 × 3	16.447	16.447	16.448	31	13	10
5 × 5	14.342	14.344	14.344	621	298	116
7 × 5	14.123	14.122	14.130	1196	740	561

Table 2 Comparison of optimal weights for x displacement at loading node ≤ 2.0 × 10^{-3} L

Problem size	Direct optimization of ground structure, lb	Sizing optimization of compliance topology, lb
4 × 3	1264	1264
5 × 5	1140	1140
7 × 5	1130	1136

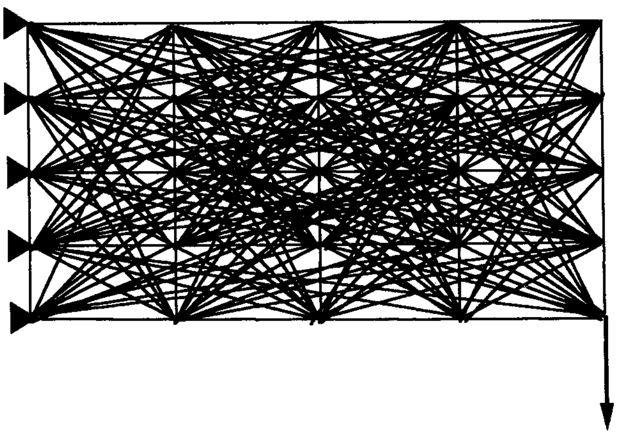

Fig. 1 Ground structure for the aspect ratio 2:1 and 5 × 5 grid.

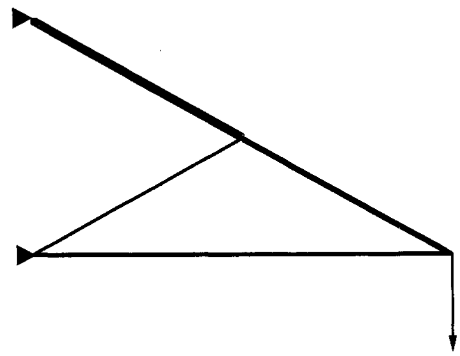

Fig. 2 Optimal truss obtained for the 4 × 3 grid with all three methods.

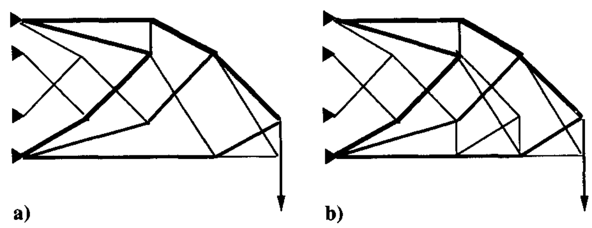

Fig. 3 Optimal trusses obtained for the 5 × 5 grid with a) compliance minimization with PF and b) weight minimization with PF and AL.

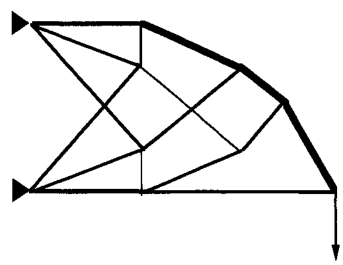

Fig. 4 Optimal truss obtained for the 7 × 5 grid with all three methods.

An extended interior penalty function[8] is used to solve this problem. The problem is reformulated as

$$\text{minimize} \quad \phi = V(x) + r\sum_{j=1}^{m} p[g_j(x, u)] + \frac{c}{\sqrt{r}} R^T B^{-1} R \qquad (3)$$

for $r = r_1, r_2, \ldots$, where $r_i \to 0$, and

$$p[g_j] = 1/g_0\left[(g_j/g_0)^2 - 3g_j/g_0 + 3\right] \qquad (4)$$

is the extended interior penalty function. The constants r and c are initially chosen to balance the contribution of each term to ϕ. The matrix B is used for preconditioning the last term, and B is an easily invertible element-by-element approximation to K. Beale's restarted conjugate gradient method[9] is used for the minimization. As the penalty parameter r is reduced the problem becomes ill conditioned and the optimization process is slowed down.

One way to avoid the ill conditioning is to use the augmented Lagrangian approach.[10,11] It adds to the objective function ϕ a term $\lambda^T R$ where λ is a vector of Lagrange multipliers. The function to be minimized becomes

$$\phi = V(x) + r\sum_{j=1}^{m} p[g_j(x, u)] + \frac{c}{\sqrt{r}} R^T B^{-1} R - \lambda^T R \qquad (5)$$

Initially λ is zero and it is updated based on the first-order necessary condition which gives a recursive formula

$$\lambda^{(k+1)} = \lambda^{(k)} - 2(c/\sqrt{r}) B^{-1} R \qquad (6)$$

With this AL approach the penalty parameter r need not be reduced to zero for the optimization to converge. In the present work the penalty parameter r was kept at the same value if the Euclidean norm of the vector R was less than 10^{-2} lbs. In the later cycles the Lagrange multipliers will take over the minimization of ϕ, the augmented penalty function.

Results

The truss topology optimization problem is to find the optimal truss to transmit an applied load to the supports as a subset of the initial ground structure. In this work the problem of finding the optimal truss to transmit a vertical load, applied at the lower right-hand corner, to the simply supported nodes on the left is solved.

The topology optimization was begun with a ground structure similar to the one shown in Fig. 1 with an aspect ratio of 2:1. The initial member areas were chosen to satisfy the stress constraints. The truss was optimized using three formulations based on the SAND approach: compliance minimization with PF formulation and weight minimization subject to stress constraints using both PF and AL formulations. The stress allowables were 25 ksi in both tension and compression. Under these conditions, the optimal designs obtained by compliance minimization should be the same as those obtained by weight minimization. The optimal truss designs obtained in each of these cases were compared with those in Ref. 7 in terms of geometry, layout, and the nondimensional compliance η defined as

$$\eta = (f^T u) VE/(\|f\|^2 L^2) \tag{7}$$

where V is the volume, E the elastic modulus, and L the horizontal length of the truss.

The nondimensional compliances and the computation times for the 2:1 aspect ratio problem are shown in Table 1. Column 1 describes the problem size. In columns 2–4 the nondimensional compliances obtained with the three methods are compared. As we increase the number of nodes in the ground structure, we achieve improvement in the minimum value of the compliance. The agreement between all three methods is good, confirming the theoretical result that weight minimization with stress constraints is equivalent to compliance minimization. Columns 5–7 compare the CPU time used by the three methods on an IBM-3090 computer. As expected, the AL approach is more efficient than the PF approach since the ill conditioning associated with a large penalty parameter has been eliminated. The large advantage of the weight minimization formulation over compliance minimization formulation may be due to programming idiosyncrasies since the results were obtained with different computer programs.

The optimal trusses obtained with these three formulations by using three different grid sizes are shown in Figs. 2–4, respectively. It is seen that all three methods predict the same optimal topology for the 4 × 3 and the 7 × 5 grid, whereas the weight minimization optimum topology for the 5 × 5 grid is different from the compliance minimum optimum. It is possible that the optimum design is not unique, or that the weight minimization result is a near optimum.

Table 3 Comparison of optimal weights (in. lb) for x displacement at loading node ≤ 1.25 × 10^{-3} L

Problem size	Direct optimization of ground structure	Sizing optimization of compliance topology
4 × 3	1433	1546
5 × 5	1274	1209
7 × 5	1259	1304

Table 4 Comparison of optimal weights (in. lb) for x displacement at loading node ≤ 0.625 × 10^{-3} L

Problem size	Direct optimization of ground structure	Sizing optimization of compliance topology
4 × 3	1648	2316
5 × 5	1448	1300
7 × 5	1441	1760

Table 5 Member elimination strategy for 2:1 geometry (compare with Table 1 to see effect on CPU time)

	Number of members			
Problem size	In-ground structure	After 5 cycles	In final design	IBM 3090 CPU time, s
4 × 3	47	6	6	10
5 × 5	196	27	27	86
7 × 5	384	102	23	222

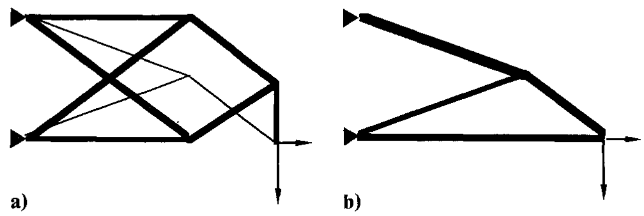

Fig. 5 Optimal trusses obtained for the 4 × 3 grid with the 50% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

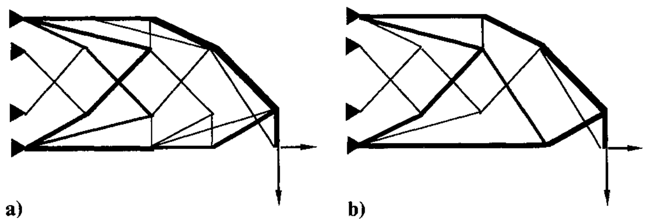

Fig. 6 Optimal trusses obtained for the 5 × 5 grid with the 50% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

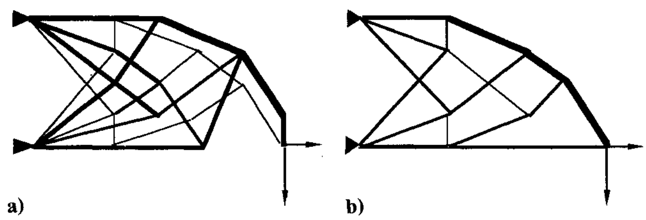

Fig. 7 Optimal trusses obtained for the 7 × 5 grid with the 50% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

Since compliance minimization can be performed very efficiently by specialized methods,[6,7] it makes sense to use it for more general problems as well, using a two-phase strategy. In the first phase we find an optimum topology by compliance minimization, and in the second phase we resize members to take care of the actual objective function and constraints. The SAND approach, on the other hand, is applicable directly to general stress and displacement constraints. The two approaches are compared for a test case where a displacement constraint is added to the existing constraints on the allowable stresses of the material used to make the truss. For the test case a ground structure with a horizontal length of 720 in. and a height of 360 in. (aspect ratio of 2:1) and a second ground structure with an aspect ratio of 8:5 were considered. All of the truss elements had an elastic modulus of 10^4 ksi and a density of 0.1 lb/in.[3] The truss was loaded with a point load of 100 kips.

The test case constraint is a constraint on the horizontal displacement at the corner (see Fig. 1) where the load is applied in addition to the stress constraints. The horizontal displacement at this node was constrained to be 0.02 L, which is 80% of the displacement at that node in the minimum compliance optimum.

The final designs obtained with both approaches are compared in Table 2. The first column describes the problem. The second column shows the optimal weight obtained by direct optimization of the ground structure. The third column gives the weight obtained by sizing optimization of the minimum compliance topology. For this displacement case it is seen that sizing optimization of the minimum compliance optimum topology is as good as direct optimization starting from the ground structure. This changed when the allowable limit on the displacement was required to be less than or equal to 50% of the unconstrained displacement. At first, the optimal designs turned out to be mechanisms since the displacement constraint is then trivially satisfied. To avoid this, a small horizontal load of 0.1 lb was additionally introduced at the lower right-hand corner, and the trusses were reoptimized. Table 3 shows that sizing optimization of the optimum compliance topology leads to a heavier design than the one obtained from direct optimization of the ground structure for the 4 × 3 grid and the 7 × 5 grid. However, for the 5 × 5 grid direct optimization yielded a heavier design compared to sizing optimization. This could be a local minimum. The optimum designs are shown in Fig. 5 for the 4 × 3 grid, in Fig. 6 for the 5 × 5 grid, and in Fig. 7 for the 7 × 5 grid. It is clear that direct optimization changed the topology to remove the horizontal member at the loading point since this makes the displacement easy to satisfy.

The allowable limit on the displacement was next set to be less than 25% of the unconstrained displacement. The trend was similar to the preceding case but the optimal trusses obtained by sizing optimization were much heavier than the SAND optima for the 4 × 3 and 7 × 5 cases. The results are shown in Table 4 and in Figs. 8–10. For this case of displacement constraint, it is seen that

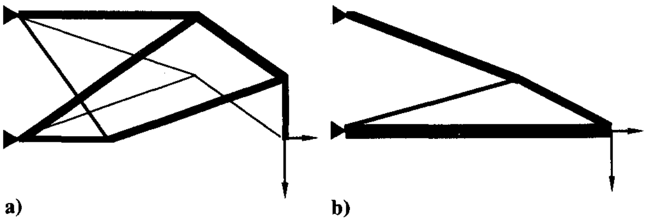

Fig. 8 Optimal trusses obtained for the 4 × 3 grid with the 25% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

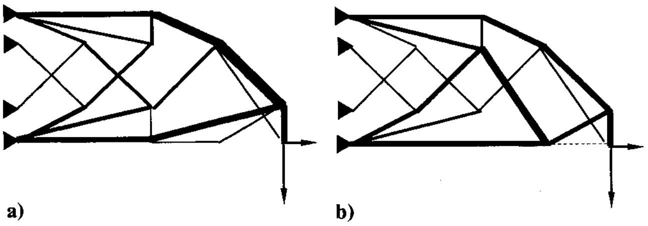

Fig. 9 Optimal trusses obtained for the 5 × 5 grid with the 25% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

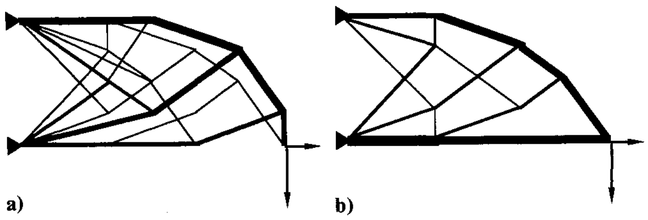

Fig. 10 Optimal trusses obtained for the 7 × 5 grid with the 25% horizontal displacement constraint at the load with a) SAND and b) sizing optimization of optimum minimum compliance topology.

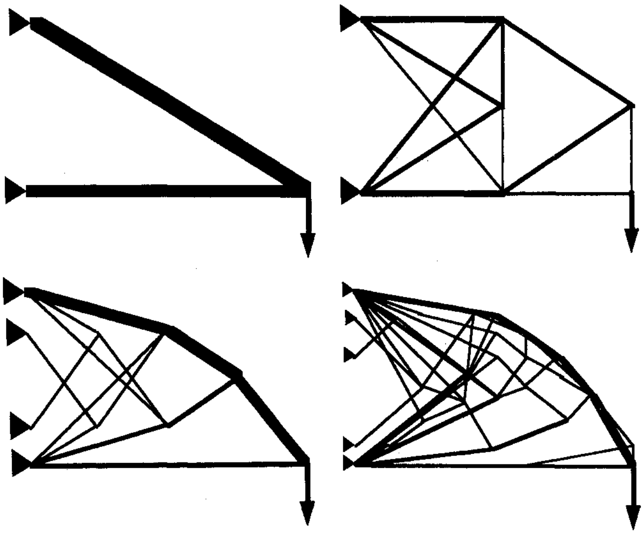

Fig. 11 Optimal topologies for the 8:5 aspect ratio domain obtained by weight minimization for a) 2 × 2 grid, b) 3 × 3 grid, c) 5 × 5 grid, and d) 9 × 9 grid.

the weight minimization of the 4 × 3 ground structure using the SAND approach and considering all of the constraints from the beginning predicted an optimal truss which weighed 40% less than the resized optimum from the compliance minimized topology.

The optimizer in a topology optimization problem reduces the cross-sectional areas of most of the members to zero. Toward the end of the optimization, a lot of time is spent in making very small cross-sectional areas even smaller.

In this work a member elimination strategy for identifying and eliminating members with small cross-sectional areas was developed and tested. After every five optimization cycles, elements with small cross-sectional areas are eliminated. An element is removed if its cross-sectional area is less than 1% of the maximum area in the current design and if simultaneously the elemental stress is less than 75% of the maximum stress. This strategy effectively weeded out most of the unimportant members. The problems described in Table 1 were optimized with the member elimination strategy applied after every five optimization cycles. The final topologies are the same, but there was considerable savings in CPU time as shown in Table 5.

The first column of Table 5 describes the problem. The second column shows the number of elements in the ground structure. The third column shows the number of members left after the elimination strategy was used once after five optimization cycles. The fourth column shows the number of members in the final optimal design. The fifth column shows the CPU time used in an IBM-3090 computer. The savings in CPU time by employing the elimination strategy are compared with those in the last column of Table 1 and can be seen to range up to 60%. It is also seen that the savings in CPU time increase with problem size.

The aspect ratio of the ground structures for all of the problems considered thus far was 1:2. Reference 7 describes the problem of transmitting a vertical force to a parallel line of supports, with a 8:5 aspect ratio grid. The same problems were solved here to check

Table 6 Effect of problem size on SAND-based algorithm with member elimination strategy for the 8 × 5 aspect ratio grid (weight minimization with AL)

	Number of members				
Problem size	In-ground structure	After 5 cycles	In final design	Nondimensional compliance	IBM 3090 CPU time, s
2 × 2	5	5	2	14.631	0.24
3 × 3	26	18	13	13.323	5.15
5 × 5	196	30	17	11.179	61
7 × 7	754	325	33	11.071	995
9 × 9	2104	205	60	10.960	2942

the effect of grid refinement on the nondimensional compliance and CPU time. The problems were solved by weight minimization using the AL formulation and the member elimination strategy. The optimal topologies are shown in Fig. 11. These are different from the 3 × 3 and the 5 × 5 topologies given in Ref. 7, but the nondimensional compliances are the same as those in Ref. 7. Table 6 shows the values obtained.

The first column of Table 6 describes the problem. The second column shows the number of members in the ground structure. Column 3 shows the number of members after five optimization cycles, and column 4 shows the number of members in the final design. Column 5 gives the nondimensional compliances for each case. Column 6 shows the CPU time in seconds for the IBM 3090. It is seen from the table that as the problem size increases, the CPU time also increases, and the nondimensional compliance decreases.

Conclusions

A SAND formulation was applied to the problem of truss topology design for minimum weight subject to stress and displacement constraints. The SAND formulation allows general displacements and stress constraints, which can be an advantage over specialized methods available for compliance minimization that may be less computationally expensive than SAND. It was demonstrated that the topology which is optimal for compliance minimization may not be optimal for a combination of stress and displacement constraints. Using an optimal compliance topology and optimizing cross-sectional areas to minimize weight was shown to result in a weight penalty of up to 40% for one case of displacement and stress constraints.

Two strategies for alleviating the computational cost of SAND approach were implemented. They are the use of an augmented Lagrangian algorithm and progessive elimination of members with small cross-sectional ares. Together these strategies reduced computational cost by up to 70%, where the bigger savings were obtained for larger problem sizes.

Acknowledgment

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